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Quaternions and ideal flows

H Eshraghi[1] **and J D Gibbon**[2]

[1] Iran University of Science and Technology (IUST), School of Physics,
Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran
[2] Department of Mathematics, Imperial College London, London SW7 2AZ, UK

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Abstract
After a review of some of the recent works by Holm and Gibbon on quaternions and their application to Lagrangian flows, particularly the incompressible Euler equations and the equations of ideal MHD, this paper investigates the compressible and relativistic Euler equations using these methods.

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1. Introduction

1.1. Background to the problem

Hamilton originally invented quaternions as an algorithm for rotating the telescope in his observatory but he was generally unable to convince his contemporaries of their importance. However, their more recent application to astro/aeronautics, robotics and computer animation has been more successful because of their efficacy in dealing with moving systems that undergo in-flight three-axis rotations (Kuipers 1999, Hanson 2006). A recent attempt has been made by Holm and co-workers to reformulate the vorticity dynamics of the three-dimensional incompressible Euler equations in terms of quaternions, particularly in tracking fluid particles that carry their own ortho-normal coordinate systems (Gibbon *et al* 2006, Gibbon and Holm 2007a, 2007b, Gibbon 2007a). For all their simplicity, the incompressible Euler equations possess subtle geometric features that are by no means understood (Majda and Bertozzi 2000, Gibbon 2007b). These properties are shared by a class of Lagrangian evolution equations of which the incompressible Euler equations and ideal MHD are just two examples (Moffatt and Tsinober 1990, 1992, Ricca and Moffatt 1992, Kuznetsov and Ruban 1998, 2000, Kuznetsov 2002). Certain variables can be naturally cast into an appropriate quaternionic form thus furnishing us with a language that is an alternative look at the dynamics of alignment processes concerning the strain matrix.

The incompressible Euler equations themselves are conventionally written in terms of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and the pressure $P(\boldsymbol{x}, t)$ as

$$\frac{D\boldsymbol{u}}{Dt} = -\rho_0^{-1}\nabla P, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla, \tag{1.1}$$

where div $\boldsymbol{u} = 0$ is the incompressibility condition and $\rho = \rho_0$ is the constant density which can be set to unity. Applying this condition to (1.1) forces the pressure to satisfy an elliptic equation

$$-\Delta P = \frac{\partial u_i}{\partial X_j}\frac{\partial u_j}{\partial X_i}, \tag{1.2}$$

the right-hand side of which involves the products of velocity gradients. Equation (1.2) can be re-written as

$$-\Delta P = \frac{\partial u_i}{\partial X_j}\frac{\partial u_j}{\partial X_i} = \mathrm{Tr}(\mathcal{S}^2) - \frac{1}{2}\omega^2 \tag{1.3}$$

in terms of the strain matrix $\mathcal{S}$

$$\mathcal{S}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right), \tag{1.4}$$

while the vorticity $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$ obeys the Euler equations in their vorticity form

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = \mathcal{S}\boldsymbol{\omega}. \tag{1.5}$$

Of course, for the compressible Euler equations, ρ is not constant but satisfies

$$\frac{D\rho}{Dt} = -\rho\,\mathrm{div}\,\boldsymbol{u}. \tag{1.6}$$

Moreover, the relation between the pressure P and ρ needs some thermodynamic input (see section 2).

The motivation for the introduction of quaternions comes from the incompressible case in the following way: define the growth and swing rate variables α and $\boldsymbol{\chi}$ as

$$\alpha = \hat{\boldsymbol{\omega}}\cdot\mathcal{S}\hat{\boldsymbol{\omega}}, \qquad \boldsymbol{\chi} = \hat{\boldsymbol{\omega}}\times\mathcal{S}\hat{\boldsymbol{\omega}}, \tag{1.7}$$

where $\hat{\boldsymbol{\omega}}$ is a unit vector along the vorticity $\boldsymbol{\omega}$. Then, using the parallel/perpendicular decomposition

$$\mathcal{S}\boldsymbol{\omega} = \alpha\boldsymbol{\omega} + \boldsymbol{\chi}\times\boldsymbol{\omega}, \tag{1.8}$$

it is easily seen that $\omega = |\boldsymbol{\omega}|$ and $\hat{\boldsymbol{\omega}}$ satisfy

$$\frac{D\omega}{Dt} = \alpha\omega, \qquad \frac{D\hat{\boldsymbol{\omega}}}{Dt} = \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}}. \tag{1.9}$$

The scalar α and the 3-vector $\boldsymbol{\chi}$ when put together as a 4-vector quaternion play a crucial role in determining the direction and growth of vorticity. The properties of these applied to the abstract Lagrangian flow and acceleration are seen in the following two subsections. Thereafter, we discuss the compressible Euler equations.

1.2. The definition and some properties of quaternions

A quaternion is constructed from a scalar s and a 3-vector $\boldsymbol{r}$ by forming the tetrad $\mathrm{q} = [s, \boldsymbol{r}]$ defined by (Tait 1890, Whittaker 1944)

$$\mathrm{q} = [s, \boldsymbol{r}] = sI - \boldsymbol{r}\cdot\boldsymbol{\sigma}^{(P)}, \tag{1.10}$$

where $\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(P)} = \sum_{i=1}^{3} r_i\sigma_i^{(P)}$ and I is the 2×2 unit matrix. $\boldsymbol{\sigma}^{(P)} = \left\{\sigma_1^{(P)}, \sigma_2^{(P)}, \sigma_3^{(P)}\right\}$ are the Pauli spin matrices

$$\sigma_1^{(P)} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_2^{(P)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_3^{(P)} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{1.11}$$

These obey the relations $\sigma_i \sigma_j = -\delta_{ij} I - \epsilon_{ijk} \sigma_k$ and generate a multiplication rule between two tetrads $\mathfrak{q}_1 = [s_1, \boldsymbol{r}_1]$ and $\mathfrak{q}_2 = [s_2, \boldsymbol{r}_2]$:

$$\mathfrak{q}_1 \circledast \mathfrak{q}_2 = [s_1 s_2 - \boldsymbol{r}_1 \cdot \boldsymbol{r}_2, s_1 \boldsymbol{r}_2 + s_2 \boldsymbol{r}_1 + \boldsymbol{r}_1 \times \boldsymbol{r}_2] . \tag{1.12}$$

A quaternion of the type $\mathfrak{w} = [0, \boldsymbol{w}]$ is called a pure quaternion. The product between two of them can be expressed as

$$\mathfrak{w}_1 \circledast \mathfrak{w}_2 = [0, \boldsymbol{w}_1] \circledast [0, \boldsymbol{w}_2] = [-\boldsymbol{w}_1 \cdot \boldsymbol{w}_2, \boldsymbol{w}_1 \times \boldsymbol{w}_2]. \tag{1.13}$$

There is also a quaternionic version of the gradient operator $\boldsymbol{\nabla} = [0, \nabla]$ which, when acting upon a pure quaternion $\mathfrak{u} = [0, \boldsymbol{u}]$, gives

$$\boldsymbol{\nabla} \circledast \mathfrak{u} = [-\mathrm{div}\, \boldsymbol{u}, \mathrm{curl}\, \boldsymbol{u}]. \tag{1.14}$$

If the field $\boldsymbol{u}$ is divergence-free, as for an incompressible fluid, then

$$\boldsymbol{\nabla} \circledast \mathfrak{u} = [0, \boldsymbol{\omega}] \equiv \mathfrak{w}, \tag{1.15}$$

which is itself a pure quaternion. $\mathfrak{w}$ will be used often in the following sections.

1.3. Lagrangian flow and acceleration

The idea has been to consider a general quaternionic picture of the process of Lagrangian flow and acceleration in fluid dynamics. This is explained in this section by considering the abstract Lagrangian flow equation[3]

$$\frac{D\boldsymbol{w}}{Dt} = \boldsymbol{\sigma}(\boldsymbol{X}, t), \tag{1.16}$$

whose Lagrangian acceleration equation is given in general by

$$\frac{D^2\boldsymbol{w}}{Dt^2} = \frac{D\boldsymbol{\sigma}}{Dt} = \boldsymbol{b}(\boldsymbol{X}, t). \tag{1.17}$$

These are the rates of change of these vectors following the characteristics of the velocity generating the path $\boldsymbol{X}(t)$ of a Lagrangian fluid particle determined from $\mathrm{d}\boldsymbol{X}/\mathrm{d}t = \boldsymbol{u}(\boldsymbol{X}, t)$ (see figure 1).

Given the Lagrangian equation (1.16), one defines the scalar α_σ and the 3-vector $\boldsymbol{\chi}_\sigma$ as

$$\alpha_\sigma = w^{-1}(\hat{\boldsymbol{w}} \cdot \boldsymbol{\sigma}), \qquad \boldsymbol{\chi}_\sigma = w^{-1}(\hat{\boldsymbol{w}} \times \boldsymbol{\sigma}), \tag{1.18}$$

in which $\boldsymbol{w} = w\hat{\boldsymbol{w}}$ with $w = |\boldsymbol{w}|$. As observed in (1.8), the 3-vector $\boldsymbol{\sigma}$ can be decomposed into parts that are parallel and perpendicular to $\boldsymbol{w}$ through the quaternionic language as

$$[0, \boldsymbol{\sigma}] = [0, \alpha_\sigma \boldsymbol{w} + \boldsymbol{\chi}_\sigma \times \boldsymbol{w}] = [\alpha_\sigma, \boldsymbol{\chi}_\sigma] \circledast [0, \boldsymbol{w}]. \tag{1.19}$$

The growth rate α_σ of the scalar magnitude $w = |\boldsymbol{w}|$ and the unit tangent vector $\hat{\boldsymbol{w}} = \boldsymbol{w} w^{-1}$ obey

$$\frac{Dw}{Dt} = \alpha_\sigma w, \qquad \frac{D\hat{\boldsymbol{w}}}{Dt} = \boldsymbol{\chi}_\sigma \times \hat{\boldsymbol{w}}, \tag{1.20}$$

as in (1.9).

This enables us to define two quaternions

$$\mathfrak{q}_\sigma = [\alpha_\sigma, \boldsymbol{\chi}_\sigma] \qquad \text{and} \qquad \mathfrak{q}_b = [\alpha_b, \boldsymbol{\chi}_b], \tag{1.21}$$

[3] In other references, the symbol $\boldsymbol{a}(\boldsymbol{X}, t)$ was used as the notation for the right-hand side of (1.16) whereas here it is $\boldsymbol{\sigma}(\boldsymbol{X}, t)$. The change of notation has been rendered necessary by the standard use of $\boldsymbol{a}$, a Lagrangian label.

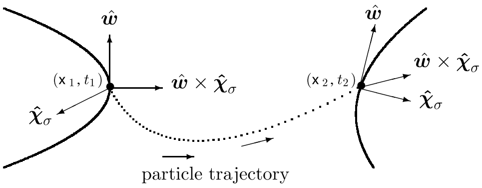

Figure 1. The dotted line represents the path of the Lagrangian fluid particle (•), whereas the solid curves represent characteristic curves to which $\boldsymbol{w}$ is a unit tangent vector. The orientation of the quaternion-frame ($\hat{\boldsymbol{w}}$, $\hat{\chi}_\sigma$, $\hat{\boldsymbol{w}} \times \hat{\chi}_\sigma$) is shown at the two spacetime points; note that this is not the Frenet-frame corresponding to the particle path but to the characteristic curves.

where α_b, $\boldsymbol{\chi}_b$ are defined as in (1.18) for α_σ, $\boldsymbol{\chi}_\sigma$ with $\boldsymbol{\sigma}$ replaced by $\boldsymbol{b}$. Let $\mathfrak{w} = [0, \boldsymbol{w}]$ be the pure quaternion satisfying the Lagrangian evolution equation (1.16). Then (1.16) can automatically be re-written in the quaternion form

$$\frac{D\mathfrak{w}}{Dt} = [0, \boldsymbol{\sigma}] = [0, \alpha_\sigma \boldsymbol{w} + \boldsymbol{\chi}_\sigma \times \boldsymbol{w}] = \mathfrak{q}_\sigma \circledast \mathfrak{w}. \tag{1.22}$$

If $\boldsymbol{\sigma}$ is Lagrangian-differentiable as in (1.17), then it is clear that a similar decomposition for $\boldsymbol{b}$ as that for $\boldsymbol{\sigma}$ in (1.8) gives

$$\frac{D^2\mathfrak{w}}{Dt^2} = [0, \boldsymbol{b}] = [0, \alpha_b \boldsymbol{w} + \boldsymbol{\chi}_b \times \boldsymbol{w}] = \mathfrak{q}_b \circledast \mathfrak{w}. \tag{1.23}$$

Using the associativity property, compatibility of (1.23) and (1.22) implies that

$$\left(\frac{D\mathfrak{q}_\sigma}{Dt} + \mathfrak{q}_\sigma \circledast \mathfrak{q}_\sigma - \mathfrak{q}_b\right) \circledast \mathfrak{w} = 0. \tag{1.24}$$

From equation (1.24), there follows the following theorem.

Theorem 1 (Gibbon and Holm 2007a, Gibbon 2007a, 2007b)**.** *The quaternions $\mathfrak{q}_\sigma$ and $\mathfrak{q}_b$ satisfy the Riccati relation*

$$\frac{D\mathfrak{q}_\sigma}{Dt} + \mathfrak{q}_\sigma \circledast \mathfrak{q}_\sigma = \mathfrak{q}_b. \tag{1.25}$$

The ortho-normal quaternion-frame $(\hat{\boldsymbol{w}}, \hat{\boldsymbol{\chi}}_\sigma, \hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_\sigma) \in SO(3)$ has Lagrangian time derivatives expressed as

$$\frac{D\hat{\boldsymbol{w}}}{Dt} = \boldsymbol{\mathcal{D}}_\sigma \times \hat{\boldsymbol{w}}, \tag{1.26}$$

$$\frac{D(\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_\sigma)}{Dt} = \boldsymbol{\mathcal{D}}_\sigma \times (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_\sigma), \tag{1.27}$$

$$\frac{D\hat{\boldsymbol{\chi}}_\sigma}{Dt} = \boldsymbol{\mathcal{D}}_\sigma \times \hat{\boldsymbol{\chi}}_\sigma, \tag{1.28}$$

where the Darboux vector $\boldsymbol{D}_\sigma$ is defined as

$$\boldsymbol{\mathcal{D}}_\sigma = \frac{c_b}{\chi_\sigma}\hat{\boldsymbol{w}} + \boldsymbol{\chi}_\sigma \qquad \text{with} \qquad c_b = \boldsymbol{\chi}_b \cdot (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_\sigma) \tag{1.29}$$

the angular frequency of rotation of the ortho-normal frame $(\hat{\boldsymbol{w}}, \hat{\boldsymbol{\chi}}_\sigma, \hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_\sigma)$.

Thus, we conclude that if we have explicit knowledge of the quartet of vectors

$$\{\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\sigma}, \boldsymbol{b}\}, \tag{1.30}$$

then the results of theorem 1 are valid. The main question, however, lies in the existence of the vector $\boldsymbol{b}$ which does not necessarily exist for every problem. In fact, Ertel's theorem is the key to this issue (Ertel 1942, Ohkitani 1993, Ohkitani and Kishiba 1995) and has been discussed at length in Gibbon and Holm (2007a, 2007b) and Gibbon (2007a, 2007b). It tells us that if $\boldsymbol{\sigma}$ takes the form $\boldsymbol{\sigma} = \boldsymbol{w} \cdot \nabla \boldsymbol{u}$, then

$$\frac{D(\boldsymbol{w} \cdot \nabla \mu)}{Dt} = \boldsymbol{w} \cdot \nabla \left(\frac{D\mu}{Dt} \right). \tag{1.31}$$

Thus, for the incompressible Euler equations, if $\boldsymbol{w} = \boldsymbol{\omega}$ and if we identify μ with the ith component of the Euler velocity field $\mu = u_i$, then

$$\boldsymbol{b} = -\mathcal{P}\boldsymbol{w}, \tag{1.32}$$

where $\mathcal{P}$ is the Hessian matrix of the pressure P:

$$\mathcal{P} = \frac{\partial^2 P}{\partial X_i \partial X_j}. \tag{1.33}$$

Thus, for the incompressible Euler equations, the quartet in (1.30) is

$$\{\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\sigma}, \boldsymbol{b}\} = \{\boldsymbol{u}, \boldsymbol{\omega}, \mathcal{S}\boldsymbol{\omega}, -\mathcal{P}\boldsymbol{w}\}. \tag{1.34}$$

For a general quartet as expressed in (1.30), $\boldsymbol{u}$ and $\boldsymbol{w}$ may be independent 3-vectors but for Euler $\boldsymbol{w} = \boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$. One must also keep in mind that $\mathcal{S}$ and $\mathcal{P}$ are also non-locally related by

$$-\operatorname{Tr} \mathcal{P} = \operatorname{Tr}(\mathcal{S}^2) - \tfrac{1}{2}\omega^2, \tag{1.35}$$

as in (1.3).

2. Compressible Euler equations with thermodynamic local equilibrium

We introduce the specific heat function (per unit mass) $h = \varepsilon + \rho^{-1}P$, where ε is the specific internal energy, P is the pressure, S is the specific entropy and ρ is the fluid density. Then we have

$$\mathrm{d}h = \mathrm{d}\varepsilon + \rho^{-1}\,\mathrm{d}P + P\,\mathrm{d}(\rho^{-1}) \tag{2.1}$$

and the first law of thermodynamics

$$\mathrm{d}\varepsilon = T\,\mathrm{d}S - P\,\mathrm{d}(\rho^{-1}). \tag{2.2}$$

Together these imply that

$$\mathrm{d}h = T\,\mathrm{d}S + \rho^{-1}\,\mathrm{d}P. \tag{2.3}$$

We can relate $\rho^{-1}\nabla P$ to ∇h and ∇S by

$$\begin{aligned} -\rho^{-1}(\nabla P \cdot \mathrm{d}\boldsymbol{X}) &= -\rho^{-1}\,\mathrm{d}P \\ &= -\mathrm{d}h + T\,\mathrm{d}S \\ &= -\nabla h \cdot \mathrm{d}\boldsymbol{X} + T\nabla S \cdot \mathrm{d}\boldsymbol{X}. \end{aligned} \tag{2.4}$$

Thus, we have

$$-\rho^{-1}\nabla P = -\nabla h + T\nabla S. \tag{2.5}$$

This can then be taken in combination with the compressible Euler equations of motion

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\rho^{-1}\nabla P, \tag{2.6}$$

the continuity equation

$$\frac{D\rho}{Dt} = -\rho\,\mathrm{div}\,\boldsymbol{u} \tag{2.7}$$

and the equation of entropy conservation

$$\frac{DS}{Dt} = 0, \tag{2.8}$$

together with an equation of state $S = S(\rho, T)$. Substitution of (2.5) into (2.6) yields

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla h + T\nabla S. \tag{2.9}$$

Using the vector identity $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \frac{1}{2}\nabla(u^2) - \boldsymbol{u}\times\boldsymbol{\omega}$ gives

$$\frac{\partial\boldsymbol{u}}{\partial t} = \boldsymbol{u}\times\boldsymbol{\omega} - \nabla\left(h + \frac{1}{2}u^2\right) + T\nabla S, \tag{2.10}$$

and curling both sides then gives

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \mathrm{curl}(\boldsymbol{u}\times\boldsymbol{\omega}) + \nabla T\times\nabla S. \tag{2.11}$$

The frozenness of $\boldsymbol{\omega}$ in the fluid motion is broken by the last term. To see this, let us use another identity

$$\mathrm{curl}(\boldsymbol{u}\times\boldsymbol{\omega}) = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{u}\,\mathrm{div}\,\boldsymbol{\omega} - \boldsymbol{\omega}\,\mathrm{div}\,\boldsymbol{u}. \tag{2.12}$$

Together with (2.7), and using the definition

$$\boldsymbol{\omega}_\rho = \rho^{-1}\boldsymbol{\omega}, \tag{2.13}$$

we have

$$\frac{D\boldsymbol{\omega}_\rho}{Dt} = \boldsymbol{\omega}_\rho\cdot\nabla\boldsymbol{u} + \rho^{-1}(\nabla T\times\nabla S). \tag{2.14}$$

Hence we have the correspondence

$$\boldsymbol{\sigma} \longleftrightarrow \boldsymbol{\omega}_\rho\cdot\nabla\boldsymbol{u} + \rho^{-1}\nabla T\times\nabla S, \tag{2.15}$$

$$\boldsymbol{w} \longleftrightarrow \boldsymbol{\omega}_\rho. \tag{2.16}$$

In terms of quaternions, we have

$$\mathfrak{q} = [\alpha_\sigma, \boldsymbol{\chi}_\sigma], \qquad \mathfrak{w} = [0, \boldsymbol{w}] \tag{2.17}$$

so specifically in terms of α_σ and $\boldsymbol{\chi}_\sigma$ defined in (1.18)

$$\alpha_\sigma = |\boldsymbol{\omega}_\rho|^{-1}\hat{\boldsymbol{\omega}}\cdot[\boldsymbol{\omega}_\rho\cdot\nabla\boldsymbol{u} + \rho^{-1}(\nabla T\times\nabla S)], \tag{2.18}$$

$$\boldsymbol{\chi}_\sigma = |\boldsymbol{\omega}_\rho|^{-1}\hat{\boldsymbol{\omega}}\times[\boldsymbol{\omega}_\rho\cdot\nabla\boldsymbol{u} + \rho^{-1}(\nabla T\times\nabla S)] \tag{2.19}$$

$$= \hat{\boldsymbol{\omega}}\times(\hat{\boldsymbol{\omega}}\cdot\nabla\boldsymbol{u}) + \omega^{-1}\hat{\boldsymbol{\omega}}\times(\nabla T\times\nabla S). \tag{2.20}$$

It is important to note that the vector $\nabla T\times\nabla S$ lies in the intersection of surfaces of constant temperature and entropy so all thermodynamical quantities are constant along this vector

$$\nabla T\times\nabla S = \left(\frac{\partial T}{\partial\rho}\nabla\rho + \frac{\partial T}{\partial S}\nabla S\right)\times\nabla S = \frac{\partial T}{\partial\rho}\nabla\rho\times\nabla S. \tag{2.21}$$

$\hat{\omega} \cdot (\nabla T \times \nabla S)$ is related to the projection of $\boldsymbol{\omega}$ on the curve of constant thermodynamical parameters. The first term in χ_σ is similar to the isentropic (barotropic) case. The second term is

$$\omega^{-1}\hat{\omega} \times (\nabla T \times \nabla S) = \rho\omega^{-2}[(\boldsymbol{\omega}_\rho \cdot \nabla S)\nabla T - (\boldsymbol{\omega}_\rho \cdot \nabla T)\nabla S]. \tag{2.22}$$

The term $\boldsymbol{\omega}_\rho \cdot \nabla S$ is a Lagrangian invariant which we will consider later.

It is worth noticing that if the equation of state is known experimentally, then it is possible to find the right-hand side of (2.22) because $\boldsymbol{u}$ and ρ can be measured experimentally. To illustrate this, let us write

$$\omega^{-1}\hat{\omega} \times (\nabla T \times \nabla S) = \rho\omega^{-2}\boldsymbol{\omega}_\rho \times \left(\frac{\partial T}{\partial \rho}\nabla\rho \times \nabla S\right) \tag{2.23}$$

$$= \rho\omega^{-2}\frac{\partial T}{\partial \rho}[(\boldsymbol{\omega}_\rho \cdot \nabla S)\nabla\rho - (\boldsymbol{\omega}_\rho \cdot \nabla\rho)\nabla S]. \tag{2.24}$$

Thus, if we can experimentally measure ρ and T as the functions of $(\boldsymbol{x}, t)$, we can evaluate χ_σ because $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are also known experimentally.

2.1. The Lagrangian coordinates

From the definitions of $\boldsymbol{\sigma}$ and $\boldsymbol{b}$ given in (1.16) and (1.17), respectively, we note that to calculate $\boldsymbol{b}$ it is better to use the Lagrangian coordinates $(\boldsymbol{a}, t)$ such that

$$\boldsymbol{u} = \frac{D\boldsymbol{X}}{Dt} = \frac{\partial \boldsymbol{X}(\boldsymbol{a}, t)}{\partial t}, \qquad \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{a}, t), \qquad \boldsymbol{a} \equiv \boldsymbol{X}(\boldsymbol{a}, t = 0). \tag{2.25}$$

So $\boldsymbol{X}(\boldsymbol{a}, t)$ is the trajectory of fluid particles. This naturally implies two kinds of spatial and time derivatives for any arbitrary fluid variable f. The first kind consists of Eulerian derivatives such as ∇f or $\partial f/\partial t$ which are taken when f is considered as a function of $\boldsymbol{X}$ and t. The other one includes Lagrangian derivatives such as $\nabla_{\boldsymbol{a}} f$ or $[\partial f(\boldsymbol{a}, t)/\partial t] = Df/Dt$ introduced when f is expressed as a function of the initial (Lagrangian) coordinates $\boldsymbol{a}$ and time t. In this paper, we frequently use these two kinds of derivatives.

Now we introduce the vector $\tilde{\boldsymbol{u}}$,

$$\tilde{\boldsymbol{u}} = u_j \nabla_{\boldsymbol{a}} X_j, \tag{2.26}$$

and define

$$\boldsymbol{\omega}_0(\boldsymbol{a}, t) = \nabla_{\boldsymbol{a}} \times \tilde{\boldsymbol{u}} = \mathrm{curl}_{\boldsymbol{a}} \tilde{\boldsymbol{u}}. \tag{2.27}$$

Then

$$\boldsymbol{\omega}_\rho = \left(\boldsymbol{\omega}_{0,\rho_0} \cdot \nabla_{\boldsymbol{a}}\right)\boldsymbol{X}, \tag{2.28}$$

$$\boldsymbol{\omega}_{0,\rho_0} \equiv \rho_0^{-1}\boldsymbol{\omega}_0 = (\boldsymbol{\omega}_\rho \cdot \nabla)\boldsymbol{a}, \tag{2.29}$$

in which $\rho_0(\boldsymbol{a})$ is the initial density distribution. In the first equation (2.28) $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{a}, t)$, and in the second equation (2.29) it can be seen that $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{X}, t)$ is indeed the inverse function of $\boldsymbol{X}(\boldsymbol{a}, t)$.

It is easily seen from (2.28) that

$$\boldsymbol{\omega}_\rho \cdot \nabla = \boldsymbol{\omega}_{0,\rho_0} \cdot \nabla_{\boldsymbol{a}}. \tag{2.30}$$

Finally, it can be shown that

$$\frac{D\boldsymbol{\omega}_0}{Dt} = \frac{\partial \boldsymbol{\omega}_0(\boldsymbol{a}, t)}{\partial t} = \nabla_{\boldsymbol{a}} T \times \nabla_{\boldsymbol{a}} S(\boldsymbol{a}). \tag{2.31}$$

The proof of equations (2.28), (2.29) and (2.31) are given in the appendix.

2.2. Ertel's theorem

Now we are able to prove a version of Ertel's theorem.

Theorem 2. *$\boldsymbol{\omega}_\rho$ and S satisfy*

$$\frac{D(\boldsymbol{\omega}_\rho \cdot \nabla S)}{Dt} = 0. \tag{2.32}$$

Proof. By making the scalar product of both sides of (2.31) with $\rho_0^{-1}\nabla_{\boldsymbol{a}} S$ and using the Lagrangian invariance of the entropy, $s = s(\boldsymbol{a})$ (see (2.8)), we find

$$\frac{\partial}{\partial t}[\boldsymbol{\omega}_{0,\rho_0}(\boldsymbol{a},t) \cdot \nabla_{\boldsymbol{a}} S(\boldsymbol{a})] = 0. \tag{2.33}$$

Finally, we apply (2.30) to (2.33) and reach to our goal. □

Now we are ready to calculate $D\boldsymbol{\sigma}/Dt = \boldsymbol{b}$ from (2.15):

$$\frac{D}{Dt}[(\boldsymbol{\omega}_\rho \cdot \nabla)\boldsymbol{u} + \rho^{-1}(\nabla T \times \nabla S)] = \frac{D}{Dt}[(\boldsymbol{\omega}_{0,\rho_0} \cdot \nabla_\sigma)\boldsymbol{u}] + \boldsymbol{u} \cdot \nabla[\rho^{-1}(\nabla T \times \nabla S)] + \frac{\partial}{\partial t}[\rho^{-1}(\nabla T \times \nabla S)]. \tag{2.34}$$

Using (2.31), the first term on the right-hand side becomes

$$\frac{\partial}{\partial t}\left[\left(\boldsymbol{\omega}_{0,\rho_0} \cdot \nabla_a\right)\boldsymbol{u}(\boldsymbol{a},t)\right] = \rho_0^{-1}[\nabla_{\boldsymbol{a}} T \times \nabla_{\boldsymbol{a}} S) \cdot \nabla_a]\boldsymbol{u}(\boldsymbol{a},t) + \left(\boldsymbol{\omega}_{0,\rho_0} \cdot \nabla_a\right)\frac{\partial \boldsymbol{u}(\boldsymbol{a},t)}{\partial t}. \tag{2.35}$$

Since $D\boldsymbol{u}/Dt = \partial \boldsymbol{u}(\boldsymbol{a},t)/\partial t$, we use (2.9) and (2.30) to calculate the second term in (2.35):

$$\begin{aligned}\frac{D}{Dt}[\boldsymbol{\omega}_\rho \cdot \nabla \boldsymbol{u}] &= \rho_0^{-1}(\nabla_{\boldsymbol{a}} T \times \nabla_{\boldsymbol{a}} S) \cdot \nabla_{\boldsymbol{a}} \boldsymbol{u} + \boldsymbol{\omega}_\rho \cdot \nabla\,(-\nabla h + T\nabla S) \\ &= \rho_0^{-1}(\nabla_{\boldsymbol{a}} T \times \nabla_{\boldsymbol{a}} S) \cdot \nabla_{\boldsymbol{a}} \boldsymbol{u} + (T\mathcal{S} - \mathcal{H})\boldsymbol{\omega}_\rho,\end{aligned} \tag{2.36}$$

where the matrices $\mathcal{S}$ and $\mathcal{H}$ are, respectively, the Hessians of S and h:

$$\mathcal{S} = \frac{\partial^2 S}{\partial X_i \partial X_j}, \qquad \mathcal{H} = \frac{\partial^2 h}{\partial X_i \partial X_j}. \tag{2.37}$$

It remains to evaluate the first term on the right-hand side of (2.36). Take the ith component of this vector

$$\rho_0^{-1}(\nabla_{\boldsymbol{a}} T \times \nabla_{\boldsymbol{a}} S) \cdot \nabla_{\boldsymbol{a}} u_i = \rho_0^{-1}\frac{\partial(u_i, T, S)}{\partial(a_1, a_2, a_3)} \tag{2.38}$$

$$= \rho_0^{-1}\nabla u_i \cdot (\nabla T \times \nabla S)\,\det\left(\frac{\partial X_k}{\partial a_j}\right). \tag{2.39}$$

Now we know the Jacobian of the transformation $\boldsymbol{a} \to \boldsymbol{X}(\boldsymbol{a},t)$ equals the ratio of the volume element d^3X to d^3a:

$$\det\left(\frac{\partial X_i}{\partial a_j}\right) = \frac{\mathrm{d}^3X}{\mathrm{d}^3a} = \frac{\rho^{-1}}{\rho_0^{-1}} = \frac{\rho_0(\boldsymbol{a})}{\rho(\boldsymbol{a},t)}. \tag{2.40}$$

Finally, we obtain

$$\rho_0^{-1}(\nabla_{\boldsymbol{a}} T \times \nabla_{\boldsymbol{a}} S)\nabla_{\boldsymbol{a}} \boldsymbol{u}(\boldsymbol{a},t) = \rho^{-1}(\nabla T \times \nabla S) \cdot \nabla \boldsymbol{u}(\boldsymbol{X},t). \tag{2.41}$$

Therefore, all terms on the right-hand side of (2.36) are known. Up to now we have

$$\boldsymbol{b} = (T\mathcal{S} - \mathcal{H})\boldsymbol{\omega}_\rho + \rho^{-1}(\nabla T \times \nabla S)\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla[\rho^{-1}(\nabla T \times \nabla S)] + \frac{\partial}{\partial t}(\rho^{-1}(\nabla T \times \nabla S)). \tag{2.42}$$

We must obtain the last term in (2.42):

$$\frac{\partial}{\partial t}[\rho^{-1}(\nabla T \times \nabla S)] = -\rho^{-2}\frac{\partial \rho}{\partial t}(\nabla T \times \nabla S) + \rho^{-1}\left[\nabla\left(\frac{\partial T}{\partial t}\right)\times\nabla S\right] + \rho^{-1}\left[\nabla T \times \nabla\left(\frac{\partial S}{\partial t}\right)\right] = \rho^{-2}\,\mathrm{div}(\rho\boldsymbol{u})(\nabla T \times \nabla S) + \rho^{-1}\nabla\left(\frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial T}{\partial S}\frac{\partial S}{\partial t}\right)\nabla S + \rho^{-1}\nabla T \times \nabla\left(\frac{\partial S}{\partial t}\right). \tag{2.43}$$

We then use entropy conservation and the continuity equation:

$$\frac{\partial}{\partial t}[\rho^{-1}(\nabla T \times \nabla S)] = \rho^{-2}\,\mathrm{div}(\rho\boldsymbol{u})(\nabla T \times \nabla S) - \rho^{-1}\nabla\left[\frac{\partial T}{\partial \rho}\mathrm{div}(\rho\boldsymbol{u}) + \frac{\partial T}{\partial S}\boldsymbol{u}\cdot\nabla S\right]\times\nabla S - \rho^{-1}\nabla T \times \nabla\,(\boldsymbol{u}\cdot\nabla S) = \rho^{-2}\mathrm{div}(\rho\boldsymbol{u})(\nabla T \times \nabla S) - \rho^{-1}\nabla\left[\boldsymbol{u}\cdot\nabla T + \rho\frac{\partial T}{\partial \rho}\,\mathrm{div}\,\boldsymbol{u}\right]\times\nabla S - \rho^{-1}\nabla T \times \nabla(\boldsymbol{u}\cdot\nabla S). \tag{2.44}$$

Thus, equations (2.43) and (2.44) determine $\boldsymbol{b}$. Then we have

$$\alpha_b = \omega_\rho^{-1}\hat{\boldsymbol{\omega}}\boldsymbol{\cdot}\boldsymbol{b}, \qquad \chi_b = \omega_\rho^{-1}\hat{\boldsymbol{\omega}}\times\boldsymbol{b}, \tag{2.45}$$

from which we can obtain the quaternion $\mathfrak{q}_b = [\alpha_b, \boldsymbol{\chi}_b]$ used in the Riccati relation

$$\frac{D\mathfrak{q}_\sigma}{Dt} + \mathfrak{q}_\sigma \circledast \mathfrak{q}_\sigma = \mathfrak{q}_b. \tag{2.46}$$

It is interesting that instead of correspondence (2.15) one may choose the Cauchy vector $\boldsymbol{\omega}_{0,\rho_0}(\boldsymbol{a}, t)$ defined in (2.28) and (2.29) which satisfies (2.31). Thus,

$$\boldsymbol{w} \longleftrightarrow \boldsymbol{\omega}_{0,\rho_0}(\boldsymbol{a}, t), \tag{2.47}$$

$$\boldsymbol{\sigma} \longleftrightarrow \rho^{-1}\nabla_{\boldsymbol{a}} T \times \nabla_{\boldsymbol{a}} S \tag{2.48}$$

may be considered in the Lagrangian coordinates $(\boldsymbol{a}, t)$ and also

$$\boldsymbol{b} = \rho_0^{-1}\nabla_{\boldsymbol{a}}\left(\frac{\partial T}{\partial t}\right)\times\nabla_{\boldsymbol{a}} S = \rho_0^{-1}\nabla_{\boldsymbol{a}}\left(\frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial t}\right)\times\nabla_{\boldsymbol{a}} S, \tag{2.49}$$

where $\partial\rho/\partial t = D\rho/Dt$.

Finally, in this section, we remark that the above formulation for the compressible case is limited to handling the inclusion of thermodynamic variables which has restricted our choices of $\boldsymbol{\sigma}$ and $\boldsymbol{b}$. These do not include the fluid variables and thus vanish when thermodynamics is excluded. A full fluid and thermodynamic formulation would be needed to explore the incompressible limit, which is our ultimate aim. The following section considers another restricted case.

3. Isentropic (barotropic) as well as incompressible Euler flow

In this section, we assume the entropy $S = \text{const}$ in the Euler equations. Instead of studying $\boldsymbol{\omega}_\rho$, we study the impulse density function

$$\boldsymbol{\gamma} = \boldsymbol{u} - \nabla\phi = \tilde{\boldsymbol{u}}_{0j}(\boldsymbol{a})\nabla a_j, \tag{3.1}$$

where $\tilde{\boldsymbol{u}}_0(\boldsymbol{a})$ is an arbitrary function fixed by initial conditions (see the appendix). The function ϕ in (3.1) is the time-dependent Bernoulli potential which satisfies

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi = \frac{1}{2}u^2 - h. \tag{3.2}$$

It is seen directly that

$$\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{\gamma}. \tag{3.3}$$

On the other hand, using relation (2.30) together with (3.1),

$$\begin{aligned}\rho^{-1}(\boldsymbol{\omega}\cdot\boldsymbol{\gamma}) &= \rho^{-1}\tilde{u}_{0j}(\boldsymbol{a})\boldsymbol{\omega}\cdot\nabla a_j \\ &= \rho_0^{-1}\tilde{u}_{0j}(\boldsymbol{a})\boldsymbol{\omega}_0\cdot\nabla_{\boldsymbol{a}} a_j \\ &= \rho_0^{-1}\tilde{u}_{0j}(\boldsymbol{a})\omega_{0,j} \\ &= \rho_0^{-1}(\boldsymbol{a})\boldsymbol{\omega}_0(\boldsymbol{a})\cdot\tilde{\mathbf{u}}_0(\boldsymbol{a}).\end{aligned} \tag{3.4}$$

Since the flow is isentropic, (2.31) reduces to

$$\frac{\partial\boldsymbol{\omega}_0(\boldsymbol{a},t)}{\partial t} = 0 \quad\Longrightarrow\quad \boldsymbol{\omega}_0 = \boldsymbol{\omega}_0(\boldsymbol{a}) \tag{3.5}$$

and so $\rho^{-1}(\boldsymbol{\omega}\cdot\boldsymbol{\gamma}) = \rho_0^{-1}(\boldsymbol{a})\boldsymbol{\omega}_0(\boldsymbol{a})\cdot\tilde{\boldsymbol{u}}_0(\boldsymbol{a})$ is a Lagrangian invariant called spirality (Eshraghi and Abedini 2005). Equation (3.2) combined with the Euler isentropic equation

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla h \tag{3.6}$$

yields

$$\begin{aligned}\frac{D\gamma_i}{Dt} &= \frac{Du_i}{Dt} - \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial X_i}\right) - \boldsymbol{u}\cdot\nabla\left(\frac{\partial\phi}{\partial X_i}\right) \\ &= -\left(\frac{\partial h}{\partial X_i}\right) - \frac{\partial}{\partial X_i}\left(\frac{D\phi}{Dt}\right) + \left(\frac{\partial u_j}{\partial X_i}\right)\left(\frac{\partial\phi}{\partial X_j}\right) \\ &= -\left(\frac{\partial h}{\partial X_i}\right) - \frac{\partial}{\partial X_i}\left(\frac{1}{2}u^2 - h\right) + \left(\frac{\partial u_j}{\partial X_i}\right)\left(\frac{\partial\phi}{\partial X_j}\right) \\ &= -u_j\frac{\partial u_j}{\partial X_i} + \left(\frac{\partial u_j}{\partial X_i}\right)\left(\frac{\partial\phi}{\partial X_j}\right) \\ &= -\frac{\partial u_j}{\partial X_i}\underbrace{\left(u_j - \frac{\partial\phi}{\partial X_j}\right)}_{\gamma_j}.\end{aligned} \tag{3.7}$$

Thus, we have

$$\frac{D\gamma_i}{Dt} = -\frac{\partial u_j}{\partial X_i}\gamma_j \quad\Longleftrightarrow\quad \frac{D\boldsymbol{\gamma}}{Dt} = -\gamma_j\nabla u_j = -(R_t R^{-1})^T\boldsymbol{\gamma}, \tag{3.8}$$

where the Jacobian matrix R is

$$R_{ij} = \frac{\partial X_i}{\partial a_j} \quad\Longrightarrow\quad R_{ij}^{-1} = \frac{\partial a_i(\boldsymbol{X},t)}{\partial X_j}, \tag{3.9}$$

and R_t is its Lagrangian time derivative

$$R_t = \frac{DR}{Dt} = \frac{\partial R(\boldsymbol{a},t)}{\partial t} \quad\Longrightarrow\quad \frac{D}{Dt}R_{ij} = \frac{\partial \dot{X}_i}{\partial a_j} = \frac{\partial u_i}{\partial a_j}. \tag{3.10}$$

The superscript T denotes the transpose of a matrix. It is possible to change the form of (3.8) by noting that

$$-\gamma_j \frac{\partial u_j}{\partial X_i} = \left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)\gamma_j - \gamma_j \frac{\partial u_i}{\partial X_j} = -\varepsilon_{ijk}\omega_k\gamma_j - \gamma_j \frac{\partial u_i}{\partial X_j} = (\boldsymbol{\omega}\times\boldsymbol{\gamma})_i - (\boldsymbol{\gamma}\cdot\nabla)u_i. \tag{3.11}$$

Therefore, (3.8) is converted to

$$\frac{D\boldsymbol{\gamma}}{Dt} = -\boldsymbol{\gamma}\cdot\nabla\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{\gamma}. \tag{3.12}$$

Such a change is possible only for the Euler equation where $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$. We are now ready for another quaternionic correspondence:

$$\boldsymbol{w} \longleftrightarrow \boldsymbol{\gamma}, \tag{3.13}$$

$$\boldsymbol{\sigma} \longleftrightarrow -\boldsymbol{\gamma}\cdot\nabla\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{\gamma}, \tag{3.14}$$

and the definitions of α_σ and χ_σ follow accordingly with

$$\mathfrak{q}_\sigma = [\alpha_\sigma, \boldsymbol{\chi}_\sigma], \qquad \mathfrak{m} = [0, \boldsymbol{\gamma}]. \tag{3.15}$$

In fact, we see that

$$[0, \rho^{-1}\boldsymbol{\omega}] \circledast [0, \boldsymbol{\gamma}] = [-\rho^{-1}(\boldsymbol{\omega}\cdot\boldsymbol{\gamma}), \rho^{-1}(\boldsymbol{\omega}\times\boldsymbol{\gamma})]. \tag{3.16}$$

From (3.4) we know that $\rho^{-1}(\boldsymbol{\omega}\cdot\boldsymbol{\gamma}) = \rho_0^{-1}(\boldsymbol{\omega}_0\cdot\boldsymbol{\gamma}_0)$ is a constant of motion and it is always possible to find *locally* a function $\tilde{\phi}(\boldsymbol{a})$ such that (Eshraghi and Abedini 2005)

$$\boldsymbol{\omega}_0\cdot\boldsymbol{\gamma}_0 = \boldsymbol{\omega}_0\cdot\boldsymbol{u}_0 - \boldsymbol{\omega}_0\cdot\nabla_{\boldsymbol{a}}\tilde{\phi} = 0, \tag{3.17}$$

and so at all times

$$\rho^{-1}(\boldsymbol{\omega}\cdot\boldsymbol{\gamma}) = 0. \tag{3.18}$$

So at least locally we can write

$$[0, \boldsymbol{\omega}] \circledast \mathfrak{m} = [0, \boldsymbol{\omega}\times\boldsymbol{\gamma}], \tag{3.19}$$

and we can write a quaternionic form of (3.12) as

$$\frac{D\mathfrak{m}}{Dt} = [0, \boldsymbol{a}] = \mathfrak{q}_\sigma \circledast \mathfrak{m} + [0, \boldsymbol{\omega}] \circledast \mathfrak{m} = (\mathfrak{q}_\sigma + [0, \boldsymbol{\omega}]) \circledast \mathfrak{m}. \tag{3.20}$$

According to (3.1), $\boldsymbol{b}$ is

$$\boldsymbol{b} = \frac{D^2}{Dt^2}[\tilde{u}_{0j}(\boldsymbol{a})\nabla a_j(\boldsymbol{X}, t)] = \tilde{u}_{0j}(\boldsymbol{a})\frac{D^2}{Dt^2}\nabla a_j, \tag{3.21}$$

which is known if the Lagrangian particle paths are known. The Riccati equation is now

$$\frac{D}{Dt}(\mathfrak{q}_\sigma + [0, \boldsymbol{\omega}]) + (\mathfrak{q}_\sigma + [0, \boldsymbol{\omega}]) \circledast (\mathfrak{q}_\sigma + [0, \boldsymbol{\omega}]) = \mathfrak{q}_b, \tag{3.22}$$

where

$$\mathfrak{q}_b = [\alpha_b, \boldsymbol{\chi}_b], \qquad \alpha_b = \gamma^{-1}(\hat{\boldsymbol{\gamma}}\cdot\boldsymbol{b}), \qquad \boldsymbol{\chi}_b = \gamma^{-1}(\hat{\boldsymbol{\gamma}}\times\boldsymbol{b}). \tag{3.23}$$

Let us simplify the Riccati equation (3.22): first, we note that

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{D(\rho\boldsymbol{\omega}_\rho)}{Dt} = \rho\boldsymbol{\omega}_\rho\cdot\nabla\boldsymbol{u} + \boldsymbol{\omega}_\rho\frac{D\rho}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} - (\operatorname{div}\boldsymbol{u})\boldsymbol{\omega}, \tag{3.24}$$

or

$$\frac{D\boldsymbol{\omega}}{Dt} = [R_t R^{-1} - Tr(R_t R^{-1})]\boldsymbol{\omega}, \tag{3.25}$$

where we have used (3.9). Next, we note that

$$[0, \boldsymbol{\omega}] \circledast \mathfrak{q}_\sigma = -[\boldsymbol{\omega} \cdot \boldsymbol{\chi}_\sigma, \alpha_\sigma \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\chi}_\sigma], \tag{3.26}$$

and

$$\mathfrak{q}_\sigma \circledast [0, \boldsymbol{\omega}] = [-\boldsymbol{\omega} \cdot \boldsymbol{\chi}_\sigma, \alpha_\sigma \boldsymbol{\omega} + \boldsymbol{\chi}_\sigma \times \boldsymbol{\omega}], \tag{3.27}$$

and so

$$[0, \boldsymbol{\omega}] \circledast \mathfrak{q}_\sigma + \mathfrak{q}_\sigma \circledast [0, \boldsymbol{\omega}] = 2[-\boldsymbol{\chi}_\sigma \cdot \boldsymbol{\omega}, \alpha_\sigma \boldsymbol{\omega}]. \tag{3.28}$$

We could also use relation (3.3) and consider $\boldsymbol{\omega}$ to be curl $\boldsymbol{\gamma}$ and eliminate the explicit appearance of $\boldsymbol{\omega}$ but we prefer to keep $\boldsymbol{\omega}$ explicitly. Finally, the Riccati equation (3.22) becomes

$$\frac{D\mathfrak{q}_\sigma}{Dt} + \mathfrak{q}_\sigma \circledast \mathfrak{q}_\sigma = \mathfrak{q}_b + \left[2\boldsymbol{\chi}_\sigma \cdot \boldsymbol{\omega} + \omega^2, -2\alpha_\sigma \boldsymbol{\omega} + (\operatorname{div} \boldsymbol{u})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}\right]. \tag{3.29}$$

4. Relativistic ideal flow

Relativistic fluids are recognized through two main properties: the fluid velocity ($v \lesssim c$) and the relativistic temperature ($k_B T \gtrsim m_0 c^2$), where m_0 is the rest mass of the fluid particles, T is the Boltzmann temperature and k_B is the Boltzmann constant. The relativistic fluid equations are

$$\frac{D\boldsymbol{p}}{Dt} = -n^{-1}\nabla P = -(\gamma n')^{-1}\nabla P, \tag{4.1}$$

$$\frac{D(\gamma n')}{Dt} = -(\gamma n') \operatorname{div} \boldsymbol{u}, \tag{4.2}$$

$$\frac{DS}{Dt} = 0, \tag{4.3}$$

which must be closed with the equation of state. Here n' is the number density in the inertial frame momentarily co-moving with the fluid, that is, the frame at which the fluid is at rest. We also have

$$\boldsymbol{p} = \frac{\gamma h}{c^2}\boldsymbol{u} \tag{4.4}$$

which is the relativistic temperature-dependent momentum and h is the heat function for each particle. All thermodynamical variables, such as the pressure P, the entropy per particle S and the heat function per particle h, are defined in the rest frame in which the number density of fluid particles is n'. We also know that

$$n = \gamma n', \qquad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}, \tag{4.5}$$

where n is the number density measured in the laboratory system.

Similar to the non-relativistic flow, we can use the thermodynamic relation $\mathrm{d}h = T\,\mathrm{d}S + n'^{-1}\,\mathrm{d}P$ to obtain

$$-\frac{\nabla P}{\gamma n'} = -\frac{\nabla h}{\gamma} + \frac{T}{\gamma}\nabla S \tag{4.6}$$

and employ it in the first equation (4.1) to find

$$\frac{D\boldsymbol{p}}{Dt} = -\frac{\nabla h}{\gamma} + \frac{T}{\gamma}\nabla S. \tag{4.7}$$

Definition (4.4) leads to

$$\begin{aligned}
\boldsymbol{u}\cdot\nabla\boldsymbol{p} &= \frac{c^2}{\gamma h}(\boldsymbol{p}\cdot\nabla)\boldsymbol{p}\\
&= \frac{c^2}{\gamma h}\left[\nabla\left(\frac{1}{2}p^2\right) - \boldsymbol{p}\times\operatorname{curl}\boldsymbol{p}\right]\\
&= \frac{1}{2\gamma h}\nabla[h^2(\gamma^2-1)] - \boldsymbol{u}\times\operatorname{curl}\boldsymbol{p}\\
&= \nabla(\gamma h) - \frac{1}{\gamma}\nabla h - \boldsymbol{u}\times\operatorname{curl}\boldsymbol{p},
\end{aligned} \tag{4.8}$$

which, when substituted into (4.7), yields

$$\frac{\partial\boldsymbol{p}}{\partial t} = \boldsymbol{u}\times\operatorname{curl}\boldsymbol{p} - \nabla(\gamma h) + \frac{T}{\gamma}\nabla S. \tag{4.9}$$

We then define the relativistic vorticity $\boldsymbol{\Omega}$

$$\boldsymbol{\Omega} \equiv \operatorname{curl}\boldsymbol{p} = \nabla\times\left(\frac{\gamma h}{c^2}\boldsymbol{u}\right) = \frac{\gamma h}{c^2}\boldsymbol{\omega} + \nabla(\gamma h)\times\frac{\boldsymbol{u}}{c^2}. \tag{4.10}$$

Thus, taking the curl of (4.9) we find

$$\frac{\partial\boldsymbol{\Omega}}{\partial t} = \operatorname{curl}(\boldsymbol{u}\times\boldsymbol{\Omega}) + \nabla\left(\frac{T}{\gamma}\right)\times\nabla S. \tag{4.11}$$

Following the same procedure as in section 2.1, we reach the dynamical relation

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\Omega}}{\gamma n'}\right) = \left(\frac{\boldsymbol{\Omega}}{\gamma n'}\cdot\nabla\right)\boldsymbol{u} + \nabla\left(\frac{T}{\gamma}\right)\times\nabla S. \tag{4.12}$$

To obtain a relativistic form of Ertel's theorem (Eshraghi 2003), similar to the non-relativistic flow, we define

$$\boldsymbol{U} \equiv p_j\nabla_a X_j(\boldsymbol{a},t) \tag{4.13}$$

where again we have used the Lagrangian variables used in (2.28) and (2.29). Consequently, we can define

$$\boldsymbol{\Omega}(\boldsymbol{a},t) = \operatorname{curl}_a\boldsymbol{U} \tag{4.14}$$

and so (see the appendix)

$$\frac{\boldsymbol{\Omega}}{n} = \left(\frac{\boldsymbol{\Omega}_0}{n_0(\boldsymbol{a})}\cdot\nabla_a\right)\boldsymbol{X}(\boldsymbol{a},t), \qquad \frac{\boldsymbol{\Omega}_0}{n_0} = \left(\frac{\boldsymbol{\Omega}}{n}\cdot\nabla\right)\boldsymbol{a}(\boldsymbol{X},t), \tag{4.15}$$

leading to

$$\frac{\boldsymbol{\Omega}}{n}\cdot\nabla = \frac{\boldsymbol{\Omega}_0}{n_0(\boldsymbol{a})}\cdot\nabla_a. \tag{4.16}$$

In the appendix, it is also shown that

$$\frac{\partial\boldsymbol{\Omega}_0}{\partial t}(\boldsymbol{a},t) = \nabla_a\left(\frac{T}{\gamma}\right)\times\nabla_a S(\boldsymbol{a}), \tag{4.17}$$

which immediately yields Ertel's theorem:

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\Omega}_0}{n_0}\cdot\nabla_a S\right)=0,\qquad \frac{D}{Dt}\left(\frac{\boldsymbol{\Omega}}{\gamma n'}\cdot\nabla S\right)=0. \tag{4.18}$$

Let us restrict ourselves to the isentropic or barotropic flow in which

$$S=\text{const}, \tag{4.19}$$

and hence $\boldsymbol{\Omega}_0=\boldsymbol{\Omega}_0(\boldsymbol{a})$ and

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\Omega}}{\gamma n'}\right)=\left(\frac{\boldsymbol{\Omega}}{\gamma n'}\cdot\nabla\right)\boldsymbol{u}. \tag{4.20}$$

We now have the correspondence

$$\boldsymbol{w}\longleftrightarrow\frac{\boldsymbol{\Omega}}{\gamma n'}, \tag{4.21}$$

$$\boldsymbol{\sigma}\longleftrightarrow\left(\frac{\boldsymbol{\Omega}}{\gamma n'}\cdot\nabla\right)\boldsymbol{u}. \tag{4.22}$$

Then

$$\alpha_\sigma=\hat{\boldsymbol{\Omega}}\cdot(\hat{\boldsymbol{\Omega}}\cdot\nabla)\boldsymbol{u},\qquad \chi_\sigma=\hat{\boldsymbol{\Omega}}\times(\hat{\boldsymbol{\Omega}}\cdot\nabla)\boldsymbol{u}. \tag{4.23}$$

Now we must calculate $\boldsymbol{b}=D\boldsymbol{\sigma}/Dt$

$$\frac{D\boldsymbol{\sigma}}{Dt}=\frac{D}{Dt}\left(\frac{\boldsymbol{\Omega}}{\gamma n'}\cdot\nabla\right)\boldsymbol{u}=\left(\frac{\boldsymbol{\Omega}}{\gamma n'}\cdot\nabla\right)\frac{D\boldsymbol{u}}{Dt}. \tag{4.24}$$

In order to calculate $D\boldsymbol{u}/Dt$, we consider equations (4.19) and (4.7)

$$\frac{D}{Dt}\left(\frac{\gamma h}{c^2}\boldsymbol{u}\right)=-\frac{\nabla h}{\gamma}, \tag{4.25}$$

from which we find

$$\frac{D\boldsymbol{u}}{Dt}=-\frac{c^2}{h^2}\frac{\nabla h}{h}-\boldsymbol{u}\left[\frac{1}{\gamma}\frac{D\gamma}{Dt}+\frac{1}{h}\frac{Dh}{Dt}\right]. \tag{4.26}$$

Even with the use of the equation of state and the continuity equation (4.1), it is unfortunately clear that, unlike non-relativistic flow, it is not possible to eliminate all time derivatives. To proceed we would need to be able to measure $D\boldsymbol{u}/Dt$ in the laboratory. In other words, quaternionic formulation would be greatly useful if we could directly measure $\boldsymbol{\sigma}$ and $\boldsymbol{b}$ in the laboratory. Then we could use them in all formulae of section 1 to find quaternionic frames and their related velocities suitable for applications. In order to measure $\boldsymbol{b}$ in the laboratory, instantaneous measurements can be made of the system (for example, the fluid or plasma). Then it is possible to provide a spatial profile of various physical parameters and therefore is not difficult to calculate 'spatial' derivatives of those parameters. Hence, if $\boldsymbol{b}$ is expressible in terms of only spatial derivatives we will be successful in using quaternions. Such a situation happens for non-relativistic fluids as was shown earlier, but unfortunately in relativistic flows we have to measure at least one 'time' derivative (see equation (4.26)). Although this makes the problem more difficult in the laboratory, nevertheless the measurement of quaternionic time derivatives is much easier than the measurement of the time derivative of other physical parameters.

As the last example, we consider the relativistic impulse density function $\boldsymbol{\Gamma}$ (see the appendix)

$$\boldsymbol{\Gamma}=\boldsymbol{p}-\nabla\Phi=U_{0j}(\boldsymbol{a})\nabla a_j(\boldsymbol{X},t), \tag{4.27}$$

where Φ is the Bernoulli potential satisfying

$$\frac{D\Phi}{Dt} = -\frac{h}{\gamma}. \tag{4.28}$$

We see that

$$\begin{aligned}\frac{D\boldsymbol{\Gamma}}{Dt} &= \frac{D\boldsymbol{p}}{Dt} - \nabla\left(\frac{D\Phi}{Dt}\right) + \frac{\partial\Phi}{\partial X_j}\nabla u_j \\ &= -\frac{\nabla h}{\gamma} + \nabla\left(\frac{h}{\gamma}\right) + \frac{\partial\Phi}{\partial X_j}\nabla u_j \\ &= h\nabla(\gamma^{-1}) + \frac{\partial\Phi}{\partial X_j}\nabla u_j.\end{aligned} \tag{4.29}$$

However,

$$\nabla(\gamma^{-1}) = -\frac{\gamma u_j}{c^2}\nabla u_j \tag{4.30}$$

$$\frac{D\boldsymbol{\Gamma}}{Dt} = \left(-\frac{\gamma h u_j}{c^2} + \frac{\partial\Phi}{\partial X_j}\right)\nabla u_j = -\Gamma_j\nabla u_j, \tag{4.31}$$

from which

$$\frac{D\boldsymbol{\Gamma}}{Dt} = -\Gamma_j\nabla u_j = -(R_t R^{-1})^T\boldsymbol{\Gamma}. \tag{4.32}$$

These equations are similar to (3.8) for γ except that here $\boldsymbol{\Omega} \neq \operatorname{curl}\boldsymbol{u}$. Hence we have the correspondence

$$\boldsymbol{w} \longleftrightarrow \boldsymbol{\Gamma}, \tag{4.33}$$

$$\boldsymbol{\sigma} \longleftrightarrow -\Gamma_j\nabla u_j, \tag{4.34}$$

with

$$\alpha_\sigma = -\hat{\boldsymbol{\Gamma}}\cdot(\hat{\boldsymbol{\Gamma}}\cdot\nabla)\boldsymbol{u}, \qquad \chi_\sigma = -\hat{\boldsymbol{\Gamma}}\times(\hat{\boldsymbol{\Gamma}}\cdot\nabla)\boldsymbol{u}. \tag{4.35}$$

Thus, if the Lagrangian paths are known, we have

$$\boldsymbol{b} = \frac{D^2\boldsymbol{\Gamma}}{Dt^2} = U_{0j}\frac{D^2}{Dt^2}\nabla a_j(\boldsymbol{X}, t), \tag{4.36}$$

from which $\mathfrak{q}_b = [\alpha_b, \chi_b]$ can be calculated.

Appendix. The Weber transformation and Lagrangian variables for the non-relativistic and relativistic cases

A.1. Non-relativistic flows

We begin with the Euler equation

$$\frac{D\boldsymbol{u}}{Dt} = \frac{D\dot{\boldsymbol{X}}}{Dt} = \ddot{\boldsymbol{X}} = -\frac{\nabla p}{\rho} = -\nabla h + T\nabla S. \tag{A.1}$$

Taking the ith component of the above equation and multiplying by $\nabla_a X_j$ to obtain

$$\ddot{X}_j\nabla_a X_j = -\nabla_a h + T\nabla_a S. \tag{A.2}$$

It is now seen that

$$\ddot{X}_j \nabla_a X_j = \frac{D}{Dt}(\dot{X}_j \nabla_a X_j) - \dot{X}_j \nabla_a \dot{X}_j = \frac{D}{Dt}(u_j \nabla_a X_j) - \frac{1}{2}\nabla_a\left(u_j^2\right). \tag{A.3}$$

With $\tilde{\boldsymbol{u}} = u_j \nabla_a X_j$ we have

$$\frac{D\tilde{\boldsymbol{u}}}{Dt} = \frac{\partial \tilde{\boldsymbol{u}}}{\partial t}(\boldsymbol{a}, t) = -\nabla_a \left(h - \frac{1}{2}u^2\right) + T\nabla_a S. \tag{A.4}$$

Taking the curl with respect to $\boldsymbol{a}$ gives

$$\frac{D\boldsymbol{\omega}_0}{Dt} = \frac{\partial \boldsymbol{\omega}_0}{\partial t}(\boldsymbol{a}, t) = \nabla_a T \times \nabla_a S, \tag{A.5}$$

where $\boldsymbol{\omega}_0 = \nabla_a \times \tilde{\boldsymbol{u}}$; now let us find the relationship between $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$ and $\boldsymbol{\omega}_0 = \nabla_a \times \tilde{\boldsymbol{u}}$:

$$\omega_{0i} = \varepsilon_{ijk}\frac{\partial \tilde{u}_k}{\partial a_j} = \varepsilon_{ijk}\frac{\partial}{\partial a_j}\left(u_\ell \frac{\partial X_\ell}{\partial a_k}\right) = \varepsilon_{ijk}\frac{\partial u_\ell}{\partial X_m}\frac{\partial X_m}{\partial a_j}\frac{\partial X_\ell}{\partial a_k} + \underbrace{\varepsilon_{ijk}u_\ell \frac{\partial^2 X_\ell}{\partial a_j \partial a_k}}_{\text{zero}}. \tag{A.6}$$

But

$$\varepsilon_{ijk} = \varepsilon_{njk}\delta_{ni} = \varepsilon_{njk}\frac{\partial X_p}{\partial a_n}\frac{\partial a_i}{\partial X_p}, \tag{A.7}$$

and so

$$\omega_{0i} = \frac{\partial u_\ell}{\partial X_m}\left(\varepsilon_{njk}\frac{\partial X_p}{\partial a_n}\frac{\partial X_m}{\partial a_j}\frac{\partial X_\ell}{\partial a_k}\right)\frac{\partial a_i}{\partial X_p}. \tag{A.8}$$

On the other hand, we know that

$$\varepsilon_{njk}\frac{\partial X_p}{\partial a_n}\frac{\partial X_m}{\partial a_j}\frac{\partial X_\ell}{\partial a_k} = \epsilon_{pm\ell} \det R, \qquad R_{ij} = \frac{\partial X_i}{\partial a_j}, \tag{A.9}$$

and therefore

$$\omega_{0i} = (\det R)\epsilon_{pm\ell}\frac{\partial u_\ell}{\partial X_m}\frac{\partial a_i}{\partial X_p}. \tag{A.10}$$

With $\omega_p = \epsilon_{pm\ell}\frac{\partial u_\ell}{\partial X_m}$, we have

$$\omega_{0i} = (\det R)\boldsymbol{\omega} \cdot \nabla a_i. \tag{A.11}$$

Thus, $\boldsymbol{\omega}_0 = (\det R)\boldsymbol{\omega} \cdot \nabla \boldsymbol{a}$. The inverse relation is also found

$$\omega_{0j}\frac{\partial X_i}{\partial a_j} = (\det R)\omega_k \frac{\partial a_j}{\partial X_k}\frac{\partial X_i}{\partial a_j} = (\det R)\omega_k \delta_{ik} = (\det R)\omega_i. \tag{A.12}$$

Therefore

$$\boldsymbol{\omega} = \frac{1}{\det R}(\boldsymbol{\omega}_0 \cdot \nabla_a)\boldsymbol{X}. \tag{A.13}$$

We have seen in section 1 that

$$\det R = \frac{\rho_0(\boldsymbol{a})}{\rho_0(\boldsymbol{a}, t)}, \tag{A.14}$$

and consequently we obtain equations (2.28) and (2.29).

In this part, we proceed to the study of the non-relativistic impulse density function $\boldsymbol{\gamma}$. We saw that in the isentropic case (A.4) reduces to

$$\frac{D\tilde{\boldsymbol{u}}}{Dt} = \frac{\partial\tilde{\boldsymbol{u}}(\boldsymbol{a},t)}{\partial t} = \nabla_a\left(\frac{1}{2}u^2 - h\right) = \nabla_a\left(\frac{D\phi}{Dt}\right) = \nabla_a\left(\frac{\partial\phi(\boldsymbol{a},t)}{\partial t}\right). \tag{A.15}$$

This means that

$$\frac{\partial\tilde{\boldsymbol{u}}(\boldsymbol{a},t)}{\partial t} = \frac{\partial\nabla_a\phi(\boldsymbol{a},t)}{\partial t}, \tag{A.16}$$

while the Bernoulli potential satisfies

$$\frac{D\phi}{Dt} = \frac{\partial\phi(\boldsymbol{a},t)}{\partial t} = \frac{1}{2}u^2 - h + f(t), \tag{A.17}$$

with $f(t)$ an arbitrary function of t which can be put to zero. Thus,

$$\frac{D\phi}{Dt} = \frac{\partial\phi(\boldsymbol{a},t)}{\partial t} = \frac{1}{2}u^2 - h. \tag{A.18}$$

Consequently,

$$\frac{\partial\tilde{\boldsymbol{u}}(\boldsymbol{a},t)}{\partial t} = \frac{\partial\nabla_a\phi(\boldsymbol{a},t)}{\partial t}, \tag{A.19}$$

which means that $\tilde{\boldsymbol{u}}(\boldsymbol{a},t) = \tilde{\boldsymbol{u}}_0(\boldsymbol{a}) + \nabla_{\boldsymbol{a}}\phi$. In turn this implies that $\boldsymbol{\omega}_0 = \mathrm{curl}_{\boldsymbol{a}}\boldsymbol{u} = \mathrm{curl}_{\boldsymbol{a}}\tilde{\boldsymbol{u}}_0(\boldsymbol{a})$ and

$$\frac{D\boldsymbol{\omega}_0}{Dt} = 0. \tag{A.20}$$

Thus, $\boldsymbol{\omega}_0 = \boldsymbol{\omega}_0(\boldsymbol{a})$. We see that at $t = 0$, $R = I$ and $\boldsymbol{X}(\boldsymbol{a},0) = \boldsymbol{a}$, so $\tilde{\boldsymbol{u}}|_{t=0} = \boldsymbol{u}|_{t=0}$ and hence at any point $\boldsymbol{X}$ in space $\mathrm{curl}_{\boldsymbol{a}}\,\tilde{\boldsymbol{u}}|_{t=0} = \mathrm{curl}\,\boldsymbol{u}|_{t=0} = \boldsymbol{\omega}|_{t=0}$. We then conclude that in the isentropic case $\boldsymbol{\omega}_0 = \mathrm{curl}_{\boldsymbol{a}}\tilde{\boldsymbol{u}} = \mathrm{curl}_{\boldsymbol{a}}\tilde{\boldsymbol{u}}(\boldsymbol{a}) = \boldsymbol{\omega}_0(\boldsymbol{a})$ is an invariant; that is, $D\boldsymbol{\omega}_0/Dt = 0$. Thus,

$$\boldsymbol{\omega}_0 = \boldsymbol{\omega}_0|_{t=0} = \boldsymbol{\omega}|_{t=0} = \text{initial vorticity}. \tag{A.21}$$

Consequently,

$$\boldsymbol{\omega}_\rho|_t = R\boldsymbol{\omega}_{0,\rho_0}(\boldsymbol{a}) = R\boldsymbol{\omega}_\rho|_{t=0}, \tag{A.22}$$

$$\boldsymbol{\omega}_{0,\rho_0} = \boldsymbol{\omega}_\rho|_{t=0} = R^{-1}\boldsymbol{\omega}_\rho|_t. \tag{A.23}$$

We multiply the j th component of $\tilde{\boldsymbol{u}} = \tilde{\mathbf{u}}_0 + \nabla_{\boldsymbol{a}}\phi$ by ∇a_j and sum over j to obtain

$$\tilde{u}_j\nabla a_j = \tilde{u}_{0j}\nabla a_j + \frac{\partial\phi}{\partial a_j}\nabla a_j \tag{A.24}$$

and use the fact that $\tilde{\boldsymbol{u}} = u_j\nabla a_j$. This means that $\boldsymbol{\gamma} = \boldsymbol{u} - \nabla\phi = \tilde{u}_{0j}(\boldsymbol{a})\nabla a_j$. Other facts about $\boldsymbol{\gamma}$ such as the invariance of $\boldsymbol{\omega}_\rho\cdot\boldsymbol{\gamma}$ and $D\gamma_j/Dt = -\gamma_j\nabla u_j$ were found in section 3.

A.2. Relativistic flows

Finally, we derive the Weber transformation for relativistic isentropic flow: for work on the Weber transformation in an Euler and Navier–Stokes context, see Constantin (2002, 2003, 2005):

$$\frac{D\boldsymbol{p}}{Dt} = \dot{\boldsymbol{p}} = -\gamma^{-1}\nabla h, \tag{A.25}$$

from which we conclude that

$$\dot{p}_j \nabla_a X_j = -\gamma^{-1} \frac{\partial h}{\partial X_j} \nabla_a X_j, \tag{A.26}$$

$$\dot{p}_j \nabla_a X_j = -\gamma^{-1} \nabla_a h. \tag{A.27}$$

But

$$\dot{p}_j \nabla_a X_j = \frac{D}{Dt} \underbrace{(p_j \nabla_a X_j)}_{\boldsymbol{U}} - p_j \nabla_a \dot{X}_j \tag{A.28}$$

$$= \frac{D\boldsymbol{U}}{Dt} - p_j \nabla_a X_j \tag{A.29}$$

$$= \frac{D\boldsymbol{U}}{Dt} + h \nabla_a (\gamma^{-1}), \tag{A.30}$$

so

$$\frac{D\boldsymbol{U}}{Dt} = -\gamma^{-1} \nabla_a h - h \nabla_a (\gamma^{-1}) \tag{A.31}$$

and therefore

$$\frac{\partial \boldsymbol{U}(\boldsymbol{a}, t)}{\partial t} = -\nabla_a (h\gamma^{-1}) = \nabla_a \left(\frac{D\Phi}{Dt} \right) = \frac{\partial \nabla_a \Phi(\boldsymbol{a}, t)}{\partial t}, \tag{A.32}$$

while

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi(\boldsymbol{a}, t)}{\partial t} = -h\gamma^{-1}. \tag{A.33}$$

Thus, $\boldsymbol{U}(\boldsymbol{a}, t) = \boldsymbol{U}_0(\boldsymbol{a}) + \nabla_\sigma \Phi$, which implies that

$$U_j \nabla a_j = U_{0j}(\boldsymbol{a}) \nabla a_j + \frac{\partial \Phi}{\partial a_j} \nabla a_j = U_{0j} \nabla a_j + \nabla \Phi. \tag{A.34}$$

However, $U_j \nabla a_j = p_k \frac{\partial X_k}{\partial a_j} \nabla a_j = p_k \nabla X_k = \boldsymbol{p}$, so

$$\boldsymbol{\Gamma} \equiv \boldsymbol{p} - \nabla \Phi = U_{0j}(\boldsymbol{a}) \nabla a_j. \tag{A.35}$$

Also there are the relativistic spirality and helicity conservations (Eshraghi 2003)

$$\frac{\boldsymbol{\Gamma} \cdot \boldsymbol{\Omega}}{n} = U_{0j}(\boldsymbol{a}) \frac{\boldsymbol{\Omega} \cdot \nabla a_j}{n} = U_{0j}(\boldsymbol{a}) \frac{\boldsymbol{\Omega}_0 \cdot \nabla_a a_j}{n_0} = \frac{\boldsymbol{\Omega}_0 \cdot \boldsymbol{U}_0}{n_0}(\boldsymbol{a}), \tag{A.36}$$

which comes about because

$$\frac{\partial \boldsymbol{U}(\boldsymbol{a}, t)}{\partial t} = -\nabla_a (h\gamma^{-1}) + T \nabla_a S, \tag{A.37}$$

and

$$\frac{\partial \boldsymbol{\Omega}_0(\boldsymbol{a}, t)}{\partial t} = \nabla_a T \times \nabla_a S. \tag{A.38}$$

For isentropic flow $\nabla_a S = 0$ and so $D\boldsymbol{\Omega}_0/Dt = 0$. Similar to the non-relativistic case

$$\frac{\boldsymbol{\Omega}}{n} = \left(\frac{\boldsymbol{\Omega}_0}{n_0} \cdot \nabla_a \right) \boldsymbol{X}, \tag{A.39}$$

which implies that

$$\frac{\boldsymbol{\Omega}_0}{n_0} = \left(\frac{\boldsymbol{\Omega}}{n} \cdot \nabla \right) \boldsymbol{a}, \tag{A.40}$$

and so

$$\frac{\boldsymbol{\Omega}}{n} \cdot \nabla = \frac{\boldsymbol{\Omega}_0}{n_0} \cdot \nabla_a. \tag{A.41}$$

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